Now let

$$y - z = (1 - z)w, \qquad dy = (1 - z)dw,$$

and

$$1 - y = (1 - z)(1 - w).$$

By substituting these results, we obtain

$$I = \int_0^1 f(z)(1 - z)^{m+n-1}dz \int_0^1 w^{m-1}(1 - w)^{n-1}dw = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)} \int_0^1 f(z)(1 - z)^{m+n-1}dz.$$

This problem is taken from Whittaker & Watson's *Modern Analysis*, p. 250. Given in *Math. Trip.*, 1894.

*Note.*—A second solution was received but no name signed to it. EDITOR.

**423. Proposed by J. B. REYNOLDS, Lehigh University.**

Show that the envelope of all circles with their centers on the circle $x^2 + y^2 = a^2$ and tangent to the $x$-axis is the two-arched epicycloid.

## SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The general form of equation of circles fulfilling the conditions of the problem is

$$f(x, y, x_1, r) = (x - x_1)^2 + (y - r)^2 - r^2 = 0, \tag{1}$$

with the conditional equation

$$\varphi(x_1, r, a) = x_1^2 + r^2 - a^2 = 0. \tag{2}$$

We are to find the envelope of the system of circles represented by (1), $x_1$, $r$ being the variable parameters. Using the undetermined multiplier $\lambda$ and the auxiliary equations

$$\frac{df}{dx_1} = \lambda \frac{d\varphi}{dx_1}, \tag{3}$$

$$\frac{df}{dr} = \lambda \frac{d\varphi}{dr}, \tag{4}$$

we have

$$-(x - x_1) = \lambda x_1, \tag{5}$$

and

$$-(y - r) - r = \lambda r. \tag{6}$$

We must eliminate $x_1$, $r$, $\lambda$ from (1), (2), (5), and (6). Substituting the values of $x - x_1$, and $y - r$ from (5) and (6) in (1),

$$\lambda^2 x_1^2 + r^2(1 + \lambda)^2 = r^2. \tag{7}$$

Also eliminating $x_1$ from (2) and (7),

$$\lambda(2r^2 + a^2\lambda) = 0. \tag{8}$$

The value of $\lambda = 0$ is irrelevant, but the second factor in (8) gives

$$r^2 = -\tfrac{1}{2}a^2\lambda. \tag{9}$$

Substituting (9) in (2) gives

$$x_1^2 = \frac{a^2}{2}(\lambda + 2) \tag{10}$$

and (5) gives

$$x_1 = \frac{x}{1 - \lambda}, \tag{11}$$

which with (10) gives the cubic in $\lambda$

$$a^2\lambda^3 - 3a^2\lambda + 2(a^2 - x^2) = 0. \tag{12}$$

Again, (6) gives

$$r = \frac{y}{\lambda}, \tag{13}$$

and this with (9) gives the second cubic in $\lambda$,

$$a^2\lambda^3 + 2y^2 = 0. \tag{14}$$

Eliminating $\lambda$ from (12) and (14) by Sylvester's method, the required envelope is given by

$$4(x^2 + y^2 - a^2)^3 - 27a^4y^2 = 0. \tag{15}$$

This is a two-cusped epicycloid, as may be shown by eliminating $\theta$ from the two parametric equations of the epicycloid, $a$ and $b$ being the radii of the fixed and generating circles,

$$x = (a + b) \cos \theta - b \cos \frac{a + b}{b} \theta, \tag{16}$$

$$y = (a + b) \sin \theta - b \sin \frac{a + b}{b} \theta, \tag{17}$$

first supposing $b = \frac{1}{2}a$.

*Note.*—Since making this solution, I discovered that this problem was proposed by Sir Arthur Cayley as No. 1812 in the *London Educational Times,* and in the year 1852.

Cayley was one of the most cordial and active contributors to the mathematical section of the *Times,* and it is interesting to notice that his problems and the most of his solutions appearing in the monthly issues of that journal have been included in his *Works.*

Problem 1812 is reproduced there, but no solution is given. Reference, however, is made to problem 1771 and its solution, his statement being that he was led to No. 1771 by his study of 1812.

The statement of 1771 is: "Given a circle and a line, it is required a find a parabola, having the line for directrix, and the circle for its circle of curvature." The solution given is rather intuitional in character, justifying the equation of the parabola required by certain tests.

Employing rectangular axes, such that $x = m$ is the given line, and $x^2 + y^2 = 1$ the given circle, the required parabola is

$$y^2 - 2\left(1 - \frac{4}{9}m^2\right)^{3/2} y + \frac{16}{27}m^3x + 1 - \frac{4}{3}m^2 = 0 \tag{$i$}$$

and its focus

$$\left\{m - \frac{8}{27}m^3, \quad \left(1 - \frac{4}{9}m^2\right)^{3/2}\right\}. \tag{$ii$}$$

Taking ($\cos \theta$, $\sin \theta$) as coördinates of a variable point on the circumference of $x^2 + y^2 = 1$, Cayley merely states that

$$x = \frac{3}{2} \cos \theta - m \cos 2\theta + \frac{1}{2} \cos 3\theta,$$

$$y = \frac{3}{2} \sin \theta - m \sin 2\theta + \frac{1}{2} \sin 3\theta,$$

are the coördinates of a variable point on the required envelope, adding what is the interesting connection of Nos. 1812 and 1771, viz., the required envelope in 1812 is a curve of the sixth order and has two cusps, which are the *foci* in the result of solution of 1771. It is to be noticed that the unreduced form of Cayley's result shows that there are *two* parabolas.

By a simple transformation of axes it may be easily shown that the polar of ($ii$) with respect to ($i$) is $x = m$, as should be.

The only other place where I have seen our problem is in the American edition of Williamson's *Differential Calculus,* 1884, but there is no certainty about the date when the problem was assigned a place in the manuscript of that text.

Also solved by Paul Capron, S. W. Reaves, H. S. Beers, I. L. Miller, William Weber, A. M. Harding, Horace Olson, G. W. Hartwell, C. P. Sousley, R. A. Johnson, F. M. Morgan, E. W. Worthington, D. F. Barrow, C. C. Yen, and the Proposer.

## MECHANICS.

Problem 332 is the same as 490 in Geometry, the solution of which appeared in the February number of the Monthly.

**333. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.**

A flywheel 21 feet in diameter makes 100 revolutions per minute. The weight of a cubic foot of its material is 448 pounds. Show that the intensity of stress on a transverse section of rim,